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Equalities between greatest common divisors involving three coprime pairs

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Abstract: A new equality of the greatest common divisor $(\gcd)$ of quantities involving three coprime pairs is proven in this note. For a_i and b_i positive integers such that $\gcd(a_i, b_i) = 1$ for $i \in \{1, 2, 3\}$ and $d_{ij} = |a_i b_j - a_j b_i|$, then $\gcd(d_{32}, d_{31}) = \gcd(d_{32}, d_{21}) = \gcd(d_{31}, d_{21})$. The proof uses properties of Farey sequences.
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1 Definitions

The Farey sequence F_N of order N is an ascending sequence of irreducible fractions between 0 and 1 whose denominators do not exceed N [1]. For example, F_5 is as follows,

$$F_5 = \left\{ \frac{0}{1}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{1}{1} \right\} .$$

The elements of the Farey sequence are called Farey fractions.

Fractions which are neighboring terms in any Farey sequence are known as a Farey pair. For example, $(0/1,\ 1/1)$ and $(3/5,\ 2/3)$ are two Farey pairs as their Farey fractions are neighbors in F_1 and F_5, respectively. For any Farey pair a_i/b_i and a_j/b_j it holds that $|a_i b_j - a_j b_i| = 1$.

The mediant of a Farey pair, a_i/b_i and a_j/b_j, is defined as $(a_i + a_j)/(b_i + b_j)$ and it is the only Farey fraction in between the Farey pair for the Farey sequence of order $b_i + b_j$. This allows to express any Farey fraction in between a Farey pair as a succession of mediant operations.

Therefore, the numerator of the Farey fraction is a linear combination of the numerators of the Farey pair fractions (similarly for the denominators). This can be illustrated for any Farey fraction h/k and for the Farey pair $(0/1,\ 1/1)$ as

$$\frac{h}{k} = \frac{(k-h)\cdot 0 + h\cdot 1}{(k-h)\cdot 1 + h\cdot 1} \; .$$

The following fundamental properties of the $\gcd$ are recalled:

$$\begin{aligned} \gcd(x,0) &= |x| \\ \gcd(mx,my) &= m\gcd(x,y), \quad m>0 \\ \gcd(x+ny,y) &= \gcd(x,y). \end{aligned}$$

2 Results

Theorem 2.1. *Let a_i and b_i be positive integers such that $\gcd(a_i,b_i)=1$ for $i\in\{1,2,3\}$. Let $d_{ij}=|a_ib_j-a_jb_i|$, then*

$$\gcd(d_{32},d_{31}) = \gcd(d_{32},d_{21}) = \gcd(d_{31},d_{21}) \; .$$

Proof. If $a_i > b_i$ for any $i\in\{1,2,3\}$ we can define new $\hat{b}_i$ as $\hat{b}_i = b_i + a_i$ for all $i\in\{1,2,3\}$ and $\hat{d}_{ij} = |a_i\hat{b}_j - a_j\hat{b}_i|$ so that $\hat{d}_{ij} = d_{ij}$ for all $i,j\in\{1,2,3\}$. Therefore it is sufficient to demonstrate Theorem 2.1 restricting to the case $a_i \le b_i$ for all $i\in\{1,2,3\}$. In this form, the three fractions a_i/b_i are Farey fractions. Furthermore, without loss of generality, we assume that the fractions are in the following order,

$$\frac{a_1}{b_1} \le \frac{a_2}{b_2} \le \frac{a_3}{b_3} \; .$$

The proof is divided in the following five cases:

1) $b_1 = b_2 = b_3$,

2) $b_1 < b_2$ and $b_1 < b_3$, which is symmetric to $b_3 < b_2$ and $b_3 < b_1$,

3) $b_2 < b_1$ and $b_2 < b_3$,

4) $b_2 = b_1 < b_3$, which is symmetric to $b_2 = b_3 \le b_1$,

5) $b_1 = b_3 < b_2$.

For case 1) let b be $b = b_1 = b_2 = b_3$ to express the three $\gcd$s of the theorem as:

$$\begin{aligned} \gcd(d_{32},d_{31}) &= b\gcd(|a_3-a_2|,|a_3-a_1|) \; , \\ \gcd(d_{32},d_{21}) &= b\gcd(|a_3-a_2|,|a_2-a_1|) \; , \\ \gcd(d_{31},d_{21}) &= b\gcd(|a_3-a_1|,|a_2-a_1|) \; . \end{aligned}$$

These three quantities are equal as can be seen by using the property $\gcd(x,y) = \gcd(x, y\pm x)$.

For case 2) $b_1 < b_2$ and $b_1 < b_3$: a_1/b_1 belongs to F_{b_1} and there exists a_1'/b_1' which forms a Farey pair with a_1/b_1 such that

$$\frac{a_1}{b_1} \le \frac{a_2}{b_2} \le \frac{a_3}{b_3} \le \frac{a_1'}{b_1'} \ .$$

As discussed in Section 1 it is possible to express a_2/b_2 and a_3/b_3 by a succession of mediant operations starting from the Farey pair a_1/b_1 and a_1'/b_1' as

$$\frac{a_2}{b_2} = \frac{(k_1-h_1)a_1 + h_1 a_1'}{(k_1-h_1)b_1 + h_1 b_1'} \quad \text{and} \quad \frac{a_3}{b_3} = \frac{(k_2-h_2)a_1 + h_2 a_1'}{(k_2-h_2)b_1 + h_2 b_1'}$$

with h_i/k_i being Farey fractions. Computing d_{ij} gives

$$d_{32} = |h_1 k_2 - h_2 k_1| \ , \quad d_{31} = h_2 \ , \quad d_{21} = h_1 \ .$$

The equalities in the theorem are obtained following the $\gcd$ properties.

For case 3) $b_2 < b_1$ and $b_2 < b_3$: a_2/b_2 belongs to F_{b_2} and there exist a_2'/b_2' and a_2''/b_2'' which form two Farey pairs with a_2/b_2 such that

$$\frac{a_2'}{b_2'} \le \frac{a_1}{b_1} \le \frac{a_2}{b_2} \le \frac{a_3}{b_3} \le \frac{a_2''}{b_2''} \ ,$$

again applying a succession of mediant operations between a_2'/b_2' or a_2''/b_2'' and a_2/b_2 yields

$$\frac{a_1}{b_1} = \frac{(k_1-h_1)a_2' + h_1 a_2}{(k_1-h_1)b_2' + h_1 b_2} \quad \text{and} \quad \frac{a_3}{b_3} = \frac{(k_2-h_2)a_2 + h_2 a_2''}{(k_2-h_2)b_2 + h_2 b_2''} \ ,$$

with h_i/k_i being Farey fractions. Computing d_{ij} gives

$$d_{32} = h_2 \ , \qquad d_{21} = k_1 - h_1 \ ,$$

$$d_{31} = |(k_1-h_1)(k_2-h_2) + h_1 h_2 + h_2(k_1-h_1)(a_2' b_2'' - a_2'' b_2')| \ .$$

The equalities in the theorem are obtained following the $\gcd$ properties.

In case 4) let b be $b = b_2 = b_1 < b_3$. The initial Farey fractions are transformed as $\frac{x}{y} \mapsto \frac{x}{y+x}$, resulting in

$$\frac{a_1}{b+a_1} < \frac{a_2}{b+a_2} < \frac{a_3}{b_3+a_3} \ .$$

This order preserving transformation within Farey fractions leaves unchanged the considered d_{ij} and since $b+a_1 < b+a_2$ and $b+a_1 < b_3+a_3$ we are in case 2), for which the proof has already been given.

In case 5) let b be $b = b_1 = b_3 < b_2$. Applying the same transformation as in case 4),

$$\frac{a_1}{b+a_1} < \frac{a_2}{b_2+a_2} < \frac{a_3}{b+a_3} \ ,$$

results in $b+a_1 < b+a_3$ and $b+a_1 < b_2+a_2$, which is case 2), for which the proof has already been given. □

References

[1] Hardy, G. H., & Wright, E. M. (1996). *An Introduction to the Theory of Numbers*, Fifth Edition, Oxford Science Publications.